Permutations avoiding a pattern from S_k and at least two patterns from S_3

Toufik Mansour

Department of Mathematics
University of Haifa, Haifa, Israel 31905
tmansur@study.haifa.ac.il

Abstract. In this paper, we find explicit formulas or generating functions for the cardinalities of the sets $S_n(T,\tau)$ of all permutations in S_n that avoid a pattern $\tau\in S_k$ and a set T, $|T|\geq 2$, of patterns from S_3. The main body of the paper is divided into three sections corresponding to the cases $|T|=2,3$ and $|T|\geq 4$. As an example, in the fifth section, we obtain the complete classification of all cardinalities of the sets $S_n(T,\tau)$ for $k=4$.

1 Introduction

Let $[k]=\{1,\dots,k\}$ be a (totally ordered) *alphabet* on k letters, and let $\alpha\in[k]^m$, $\beta\in[l]^m$ with $l\leq k$. We say that α is *order-isomorphic* to β if the following condition holds for all $1\leq i<j\leq n$: $\alpha_i<\alpha_j$ if and only if $\beta_i<\beta_j$.

We say that $\tau\in S_n$ *contains* $\alpha\in S_k$ if there exist $1\leq i_1<\ldots<i_k\leq n$ such that $(\tau_{i_1},\dots,\tau_{i_k})$ is order-isomorphic to $\alpha=(\alpha_1,\dots,\alpha_k)$, and we say that τ *avoids* α if τ does not contain α. The set of all permutations in S_n avoiding α is denoted $S_n(\alpha)$. More generally, for any finite set of permutations T we write $S_n(T)$ to denote the set of permutations in S_n avoiding all the permutations in T. Two sets, T_1,T_2, are said to be *Wilf equivalent* (or to belong to the same *Wilf class*) if and only if $|S_n(T_1)|=|S_n(T_2)|$ for any $n\geq 0$; the Wilf class of T we denote by $\overline{T}$.

The study of the sets $S_n(\alpha)$ was initiated by Knuth [5], who proved that $|S_n(\alpha)|=\frac{1}{n+1}\binom{2n}{n}$ for any $\alpha\in S_3$. Knuth's results where further extended in two directions. West [8] and Stankova [7] analyzed $S_n(\alpha)$ for $\alpha\in S_4$ and obtained the complete classification, which contains 3 distinct Wilf classes. This classification, however, does not give exact values of $S_n(\alpha)$. On the other hand, Simion and Schmidt [6] studied $S_n(T)$ for arbitrary subsets $T\subseteq S_3$ and discovered 7 Wilf classes. The study of $S_n(\alpha,\tau)$ for all $\alpha\in S_3$, $\tau\in S_4(\alpha)$, was completed by West [8], Billey, Jockusch and Stanley [1], and Guibert [4].

In the present paper, we calculate the cardinalities of the sets $S_n(T,\tau)$ for all $T\subseteq S_3$, $|T|\geq 2$, and all permutations $\tau\in S_k$ such that $k\geq 3$.

Remark 1. West [8] observed that if τ contains a pattern in T, then $|S_n(T,\tau)| = |S_n(T)|$. Therefore, in what follows we assume that $\tau \in S_k(T)$.

Throughout the paper, we often make use of the following simple statement.

Lemma 1. *Let $\{s_i(x)\}_{i=1}^r$, $\{A_i(x)\}_{i=1}^r$ and $\{B_i(x)\}_{i=1}^r$ be sequences of functions such that*

$$s_i(x) = A_i(x)s_{i+1}(x) + B_i(x),$$

where $1 \le i \le r-1$, and $s_r(x) = h(x)$. Then

$$s_1(x) = \begin{vmatrix} B_1(x) & -A_1(x) & 0 & \dots & 0 \\ B_2(x) & 1 & -A_2(x) & \dots & 0 \\ B_3(x) & 0 & 1 & \dots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ B_{r-1}(x) & 0 & 0 & \ddots & -A_{r-1}(x) \\ h(x) & 0 & 0 & \dots & 1 \end{vmatrix}.$$

Proof. Immediately, by definitions and induction on r. ∎

Our calculation is divided into three sections corresponding to the cases $|T| = 2, 3$ and $|T| \ge 4$. In the last section, as an example, we will obtain the complete classification (Table 1) of all cardinalities of the sets $S_n(T,\tau)$ where $T \subseteq S_3$, $\tau \in S_4$.

2 Avoiding a pair from S_3 and a pattern from S_k

In this section, we calculate the cardinality of the sets $S_n(\beta^1, \beta^2, \tau)$ where $\beta^1, \beta^2 \in S_3$, $\tau \in S_k$, $k \ge 3$. By Remark 1 and by the three natural operations, the complementation, the reversal and the inverse (see Simion and Schmidt [6], Lemma 1), we have to consider the following four possibilities:

1) $S_n(123, 132, \tau)$, where $\tau \in S_k(123, 132)$,
2) $S_n(123, 231, \tau)$, where $\tau \in S_k(123, 231)$,
3) $S_n(132, 213, \tau)$, where $\tau \in S_k(132, 213)$,
4) $S_n(213, 231, \tau)$, where $\tau \in S_k(213, 231)$.

The main body of this section is divided into four subsections corresponding to the above four cases.

2.1 $T = \{123,\ 132\}$.

Let $a_\tau(n) = |S_n(123, 132, \tau)|$, and let $a_\tau(x)$ be the generating function of the sequence $a_\tau(n)$, that is,

$$a_\tau(x) = \sum_{n \ge 0} a_\tau(n) x^n.$$

We find an explicit expression for the generating function $a_\tau(x)$.

Theorem 1. *Let $\tau \in S_k(123, 132)$. Then:*

(i) there exist $r_1, r_2, \dots, r_m \geq 1$ with $r_1 + r_2 + \ldots + r_m = k$ such that

$$\tau = (\beta_1, \beta_2, \dots, \beta_m),$$

where $\beta_i = (t_i - 1, t_i - 2, \dots, t_i - r_i + 1, t_i)$, and $t_i = k - (r_1 + \ldots + r_{i-1})$ for $i = 1, 2, \dots, m$;

(ii)

$$a_\tau(x) = \begin{vmatrix} f_{r_1}(x) & -g_{r_1}(x) & 0 & \dots & 0 \\ f_{r_2}(x) & 1 & -g_{r_2}(x) & \ddots & 0 \\ f_{r_3}(x) & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -g_{r_{m-1}}(x) \\ f_{r_m}(x) & 0 & 0 & 0 & 1 \end{vmatrix},$$

where $f_d(x) = \frac{1-x}{1-2x+x^d}$ and $g_d(x) = \frac{x^d}{1-2x+x^d}$.

Proof. (i) Let $\tau \in S_k(123, 132)$; choose r_1 such that $\tau_{r_1} = k$. Since τ avoids 132, we see that $\tau_j \geq k - r_1 + 1$ for all $j \leq r_1$, and since τ avoids 123, we get that $\tau = (\beta_1, \tau')$, where $\tau' \in S_{k-r_1}(123, 132)$, and so on.

(ii) Let $\tau = (\beta_1, \dots, \beta_m)$, and let $\alpha \in S_n(123, 132, \tau)$; choose t such that $\alpha_t = n$. Similarly to (i), $\alpha = (n-1, \dots, n-t+1, n, \alpha_{t+1}, \dots, \alpha_n)$, therefore

$$a_\tau(n) = \sum_{t=1}^{r_1-1} a_\tau(n-t) + \sum_{t=r_1}^{n} a_{\tau'}(n-t),$$

which means

$$a_\tau(n) = 2a_\tau(n-1) - a_\tau(n-r_1) + a_{\tau'}(n-r_1)$$

for all $n \geq k+1$, where $\tau' = (\beta_2, \dots, \beta_m)$. Hence

$$a_{(\beta_i,\dots,\beta_m)}(n) = 2a_{(\beta_i,\dots,\beta_m)}(n-1) - a_{(\beta_i,\dots,\beta_m)}(n-r_i) + a_{(\beta_{i+1},\dots,\beta_m)}(n-r_i)$$

for all $n \geq t_i + 1$, or equivalently,

$$\begin{aligned} \sum_{n \geq t_i+1} a_{(\beta_i,\dots,\beta_m)}(n)x^n &= 2x \sum_{n \geq t_i} a_{(\beta_i,\dots,\beta_m)}(n)x^n \\ &\quad -x^{r_i} \sum_{n \geq t_i+1} a_{(\beta_i,\dots,\beta_m)}(n)x^n \\ &\quad +x^{r_i} \sum_{n \geq t_i+1} a_{(\beta_{i+1},\dots,\beta_m)}(n)x^n. \end{aligned}$$

Since $a_{(\beta_i,\dots,\beta_m)}(n) = 2^{n-1}$ for all $n \leq t_i - 1$, $a_{(\beta+i,\dots,\beta_m)}(t_i) = 2^{t_i-1} - 1$, and $a_{(\beta_i,\dots,\beta_m)}(0) = 1$ (Simion and Schmidt [6], Proposition 7), we obtain that

$$(1 - 2x + x^{r_i})a_{(\beta_i,\dots,\beta_m)}(x) = 1 - x + x^{r_i} a_{(\beta_{i+1},\dots,\beta_m)}(x)$$

for all $i \leq m-1$, and

$$(1-2x+x^{r_m})a_{(\beta_m)}(x) = 1-x.$$

Hence, by Lemma 1, the theorem holds. ∎

Example 1. Let $T = \{123, 132\}$. By Theorem 1,

1. $|S_n(T, 3214)| = t_n$, where t_n is the n-th Tribonacci number [3], and $|S_n(T, 3241)| = f_{n+2} - 1$, where f_n is the n-th Fibonacci number.

2. $|S_n(T, 3412)| = |S_n(T, 4231)| = \binom{n}{2} + 1$.

3. $|S_n(T, 3421)| = 3n - 5$.

2.2 $T = \{123, 231\}$.

In this subsection, we calculate the cardinality of the set $S_n(123, 231, \tau)$, where $\tau \in S_k(123, 231)$. This cardinality we denote by $b_\tau(n)$.

Lemma 2. *Let $\tau \in S_k(123, 231)$. Then, either there exists r, $1 \leq r \leq k-1$, such that $\tau = (r, \ldots, 2, 1, k, k-1, \ldots, r+1)$, and hence*

$$b_\tau(n) = (k-2)n - \frac{k(k-3)}{2} \quad \textit{for all } n \geq k,$$

or $\tau = (k, \tau') \neq (k, \ldots, 2, 1)$ such that $\tau' \in S_{k-1}(123, 231)$, and hence

$$b_\tau(n) = b_{\tau'}(n-1) + n - 1 \quad \textit{for all } n \geq k.$$

Proof. Let $\tau \in S_k(123, 231)$; put $r = \tau_1$. Since τ avoids 123, we see that τ contains $(r, k, \ldots, r+1)$, since τ avoids 231, we get that $\tau = (r, \tau', k, k-1, \ldots, r+1)$, and since τ avoids 123, we have two cases: either $\tau = (r, \ldots, 1, k, \ldots, r+1)$ for $1 \leq r \leq k-1$, or $\tau = (k, \tau')$ such that $\tau' \in S_{k-1}(123, 231)$. Now let us consider the two cases:

1. Let $\alpha \in S_n(123, 231, \tau)$, where $\tau = (r, \ldots, 1, k, \ldots, r+1)$, $1 \leq r \leq k-1$. Similarly to the above, we have two cases for α: in the first case $\alpha = (t, \ldots, 1, n, n-1, \ldots, t+1)$ for $1 \leq t \leq n-1$, so there are $k-2$ permutations like α. In the second case $\alpha = (n, \alpha_2, \ldots, \alpha_n)$, so there are $b_\tau(n-1)$ permutations, which means $b_\tau(n) = b_\tau(n-1) + k - 2$. Besides, $a_\tau(k) = k(k-1)/2$ (see Simion and Schmidt [6], Proposition 11), hence $b_\tau(n) = (k-2)n - \frac{k(k-3)}{2}$.
2. Let $\alpha \in S_n(123, 231, \tau)$, and let $\tau = (k, \tau') \neq (k, \ldots, 1)$ such that $\tau' \in S_{k-1}(123, 231)$. Similarly to the above, we have two cases for α: in the first case $\alpha = (t, \ldots, 1, n, n-1, \ldots, t+1)$ for $1 \leq t \leq n-1$, so there are $n-1$ permutations like α. In the second case $\alpha = (n, \alpha_2, \ldots, \alpha_n)$, so there are $b_{\tau'}(n-1)$ permutations. Hence $b_\tau(n) = b_{\tau'}(n-1) + n - 1$.

■

Theorem 2. *Let $\tau \in S_k(123, 231)$. Then :*

(i) there exist m, $2 \le m \le k+1$, and r, $1 \le r \le m-2$, such that

$$\tau = (k, \dots, m, r, \dots, 1, m-1, \dots, r+1);$$

(ii) for all $n \ge k$

$$b_\tau(n) = (k-2)n - \frac{k(k-3)}{2}.$$

Proof. (i) Immediately, by Lemma 2,

$$\tau = (k, \dots, m, r, \dots, 1, m-1, \dots, r+1),$$

where $2 \le m \le k+1$, $1 \le r \le m-2$.

(ii) Again, by Lemma 2, for all $n \ge k$

$$b_\tau(n) = \sum_{j=1}^{k-m+1} (n-j) + (m-3)(n-(k-m+1)) + \frac{(m-1)(4-m)}{2},$$

hence, this theorem holds. ■

Example 2. Let $T = \{123, 231\}$; by the Theorem 2

$$|S_n(T, 4312)| = |S_n(T, 1432)| = |S_n(T, 2143)| = |S_n(T, 3214)| = 2n-2.$$

2.3 $T = \{132, 213\}$.

Let $c_\tau(n) = |S_n(132, 213, \tau)|$, and let $c_\tau(x)$ be the generating function of the sequence $c_\tau(n)$. We find an explicit expression for the generating function $c_\tau(x)$.

Theorem 3. *Let $\tau \in S_k(132, 213)$. Then:*

(i) there exist $k+1 = r_0 > r_1 > \ldots > r_m \ge 1$ such that

$$\tau = (r_1, r_1+1, \dots, k, r_2, r_2+1, \dots, r_1-1, \dots, r_m, r_m+1, \dots, r_{m-1}-1);$$

(ii)

$$c_\tau(x) = \begin{vmatrix} f_{r_0-r_1}(x) & -g_{r_0-r_1}(x) & 0 & \dots & 0 \\ f_{r_1-r_2}(x) & 1 & -g_{r_1-r_2}(x) & \ddots & 0 \\ f_{r_2-r_3}(x) & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -g_{r_{m-2}-r_{m-1}}(x) \\ f_{r_{m-1}-r_m}(x) & 0 & 0 & 0 & 1 \end{vmatrix},$$

where $f_d(x)$ and $g_d(x)$ have the same meaning as in Theorem 1.

Proof. (i) Let $\tau \in S_k(132,213)$, and let $r_1 = \tau_1$. Since τ avoids 132, we see that τ contains $(r_1, r_1+1, \dots, k)$, and since τ avoids 213, we get that $\tau = (r_1, r_1+1, \dots, k, \tau')$, where $\tau' \in S_{r_1-1}(132,213)$, and so on.

(ii) Let $\alpha \in S_n(132,213,\tau)$, and let $t = \alpha_1$; similarly to (i), $\alpha = (t, t+1, \dots, n, \alpha_{n-t+2}, \dots, \alpha_n)$. Therefore, for $t \le n-k+r_1$ we have $\alpha \in S_n(132,213,\tau)$ if and only if $(\alpha_{n-t+2}, \dots, \alpha_n) \in S_{t-1}(132,213,\tau')$, and for $t \ge n-k+r_1+1$ we have $\alpha \in S_n(132,213,\tau)$ if and only if $(\alpha_{n-t+2}, \dots, \alpha_n) \in S_{t-1}(132,213,\tau)$. Hence

$$c_\tau(n) = \sum_{t=1}^{n-k+r_1} c_{\tau'}(t-1) + \sum_{t=n-k+r_1+1}^{n} c_\tau(t-1),$$

which means that

$$c_\tau(n) = 2c_\tau(n-1) - c_\tau(n-k+r_1-1) + c_{\tau'}(n-k+r_1-1).$$

Let us define $t_i = r_{i-1} - r_i$ for $i = 1, 2, \dots, m$, so

$$c_\tau(n) = 2c_\tau(n-1) - c_\tau(n-t_1) + c_{\tau'}(n-t_1).$$

If $\tau = (1,2,\dots,k)$, then immediately $c_\tau(x) = \frac{1-x}{1-2x+x^k}$, hence, by Lemma 1, this theorem holds. ■

Corollary 1. *Let $k \ge 2$. For all $n \ge 0$,*

$$c_{(k,\dots,2,1)}(n) = \sum_{j=0}^{k-2} \binom{n-1}{j}.$$

Proof. By the proof of Theorem 3,

$$c_{(k,\dots,2,1)}(n) = 2^k + \sum_{t=k-1}^{n-1} c_{(k-1,\dots,2,1)}(t),$$

which means that

$$c_{(k,\dots,2,1)}(n) = c_{(k,\dots,2,1)}(n-1) + c_{(k-1,\dots,2,1)}(n-1).$$

Besides, $c_{(k,\dots,2,1)}(k) = 2^{k-1} - 1$, and $c_{(k,\dots,2,1)}(n) = 2^{n-1}$ for $1 \le n \le k-1$ (see Simion and Schmidt [6], Proposition 8). Hence, the corollary is true. ■

Example 3. Let $T = \{132, 213\}$.

1. By Corollary 1, $|S_n(T, 4321)| = \binom{n}{2} + 1$.

2. By Theorem 3, $|S_n(T, 1234)| = t_n$ where t_n is the n-th Tribonacci number.

3. By Theorem 3, $|S_n(T, 2341)| = f_{n+2} - 1$ where f_n is the n-th Fibonacci number.

4. By Theorem 3, $|S_n(T, 3412)| = |S_n(T, 3421)| = |S_n(T, 4231)| = \binom{n}{2} + 1$.

2.4 $T = \{213,\ 231\}$.

Let $d_\tau(n) = |S_n(213, 231, \tau)|$, and let $d_\tau(x)$ be the generating function of the sequence $d_\tau(n)$. We find an explicit expression for the generating function $d_\tau(x)$.

Theorem 4. *Let $\tau \in S_k(213, 231)$. Then:*

(i) τ_i is either the right maximum, or the righr minimum, for all $1 \le i \le k-1$;
(ii)

$$d_\tau(x) = \begin{vmatrix} 1 & -g(x) & 0 & \dots & 0 \\ 1 & 1 & -g(x) & \ddots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & 0 & 0 & -g(x) \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix},$$

where $g(x) = \frac{x}{1-x}$.

Proof. (i) Let $\tau \in S_k(213, 231)$; if $2 \le \tau_1 \le k-1$, then τ contains either $(\tau_1, 1, k)$ or $(\tau_1, k, 1)$, which means τ contains either 213 or 231, hence $\tau_1 = 1$ or $\tau_1 = k$, and so on.

(ii) Let $\alpha \in S_n(213, 231, \tau)$; similarly to (i), $\alpha_1 = 1$ or $\alpha_1 = n$. Let $\tau = (\tau_1, \tau')$, hence in the above two cases ($\tau_1 = 1$ or $\tau_1 = k$) we obtain

$$d_\tau(n) = d_\tau(n-1) + d_{\tau'}(n-1)$$

for all $n \ge k$. Besides, $d_\tau(0) = 1$, $d_\tau(k) = 2^{k-1} - 1$, and $d_\tau(n) = 2^{n-1}$ for all $1 \le n \le k-1$ (see Simion and Schmidt [6], Proposition 10). Hence, similarly to Theorem 1, the theorem holds. ∎

Immediately by Theorem 4,

$$|S_n(123, 132, 231)| = |S_n(132, 231, 321)| = n,$$

for all $n \ge 0$, which means, we have a generalization Proposition 16 and Lemma $6(b)$ of Simion and Schmidt [6].

Example 4. Let $T = \{213, 231\}$. By Theorem 4,

$$|S_n(T, 1234)| = |S_n(T, 1243)| = |S_n(T, 1423)| = |S_n(T, 1432)| = \binom{n}{2} + 1.$$

3 Three patterns from S_3 and a pattern from S_k

In this section, we calculate the cardinality of the sets $S_n(T,\tau)$ such that $T \subset S_3$, $|T| = 3$ and $\tau \in S_k(T)$ for $k \geq 3$. By Remark 1 and by three natural operations the complementation, the reversal and the inverse (see Simion and Schmidt [6], Lemma 1), we have to consider the following five possibilities:

1) $S_n(123, 132, 213, \tau)$, where $\tau \in S_k(123, 132, 213)$,
2) $S_n(123, 132, 231, \tau)$, where $\tau \in S_k(123, 132, 231)$,
3) $S_n(123, 213, 231, \tau)$, where $\tau \in S_k(123, 213, 231)$,
4) $S_n(123, 231, 312, \tau)$, where $\tau \in S_k(123, 231, 312)$,
5) $S_n(132, 213, 231, \tau)$, where $\tau \in S_k(132, 213, 231)$.

Remark 2. By Erdös and Szekeres [2], $|S_n((1,2,\dots,a),(b,b-1,\dots,1))| = 0$ for all $n \geq (a-1)(b-1)+1$, where $a, b \geq 1$. Therefore, in what follows we assume that $\tau \in S_k(T)$ and $\tau \neq (k, k-1, \dots, 1)$, since $123 \in T$.

The main body of this section is divided into five subsections corresponding to the above five cases.

3.1 $T = \{123, 132, 213\}$.

Let $e_\tau(x)$ be the generating function of the sequence $|S_n(T,\tau)|$. We find an explicit expression for the generating function $e_\tau(x)$.

Lemma 3. *Let $T = \{123, 132, 213\}$ and $\tau \in S_k(T)$. Then, either there exists $\tau' \in S_{k-1}(T)$ such that $\tau = (k, \tau') \neq (k, k-1, \dots, 1)$, and hence*

$$|S_n(T,\tau)| = |S_{n-1}(T,\tau')| + |S_{n-2}(T,\tau')| \quad \textit{for any } n \geq k,$$

or there exists $\tau'' \in S_{k-2}(T)$ such that $\tau = (k-1, k, \tau')$, and hence

$$|S_n(T,\tau)| = |S_{n-1}(T,\tau)| + |S_{n-2}(T,\tau'')| \quad \textit{for any } n \geq k.$$

Proof. Let $\tau \in S_k(T)$; since τ avoids 123 and 132 we have either $\tau_1 = k$ or $\tau_1 = k-1$. If $\tau_1 = k-1$, then, since τ avoids 213, we see that $\tau = (k-1, k, \tau'')$. Now we consider the two cases:

1. Let $\tau = (k, \tau')$, $\alpha \in S_n(T,\tau)$. Similarly to the above, either $\alpha = (n, \alpha_2, \dots, \alpha_n)$, or $\alpha = (n-1, n, \alpha_3, \dots, \alpha_n)$, so evidently
$$|S_n(T,\tau)| = |S_{n-1}(T,\tau')| + |S_{n-2}(T,\tau')|.$$
2. Let $\tau = (k-1, k, \tau'')$, $\alpha \in S_n(T,\tau)$. Similarly to the above, either $\alpha = (n, \alpha_2, \dots, \alpha_n)$, or $\alpha = (n-1, n, \alpha_3, \dots, \alpha_n)$, so evidently
$$|S_n(T,\tau)| = |S_{n-1}(T,\tau)| + |S_{n-2}(T,\tau'')|.$$

∎

For any permutation $\tau \in S_k$ such that $\tau_1 = k$ we define $p(\tau) = 1$ and $q(\tau) = (\tau_2, \ldots, \tau_k)$, and for any permutation $\tau \in S_k$ such that $\tau_1 = k-1$, and $\tau_2 = k$ we define $p(\tau) = 2$ and $q(\tau) = (\tau_3, \ldots, \tau_k)$. Also, let $m(\tau) = (m_1, m_2, \ldots, m_r)$ where $m_i = p(q^{i-1}(\tau)))$ for $1 \leq i \leq r$, and $q^0(\tau) = \tau$, $q^{i-1}(\tau) = q(q^{i-2}(\tau)$ for $i \geq 2$.

Theorem 5. *Let $T = \{123, 132, 213\}$, $\tau \in S_k(T)$, and $m(\tau) = (m_1, \ldots, m_r)$. Then*

$$e_\tau(x) = \begin{vmatrix} u_{m_1}(x) & -v_{m_1}(x) & 0 & \ldots & 0 \\ u_{m_2}(x) & 1 & -v_{m_2}(x) & \ddots & 0 \\ u_{m_3}(x) & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -v_{m_{r-1}}(x) \\ u_{m_r}(x) & 0 & 0 & 0 & 1 \end{vmatrix},$$

where $u_1(x) = 1$, $u_2(x) = \frac{1}{1-x}$, $v_1(x) = x(1+x)$, and $v_2(x) = \frac{x^2}{1-x}$.

Proof. Let $\tau \in S_k(T)$; by Lemma 3, there are two cases:

1. $\tau_1 = k$. So $p(\tau) = 1$, $q(\tau) = (\tau_2, \ldots, \tau_k)$, and for all $n \geq k$

   $$|S_n(T, \tau)| = |S_{n-1}(T, \tau^2)| + |S_{n-2}(T, \tau^2)|.$$

   Besides, $|S_n(T, \tau)| = f_n$ for all $n \leq k-1$ (see Simion and Schmidt [6], Proposition 15), where f_n is the n-th Fibonacci number. Hence

   $$e_\tau(x) = x(1+x)e_{q(\tau)}(x) + 1.$$

2. $\tau_1 = k-1$. So $p(\tau) = 2$, $q(\tau) = (\tau_3, \ldots, \tau_k)$, and for all $n \geq k$,

   $$|S_n(T, \tau)| = |S_{n-1}(T, \tau)| + |S_{n-2}(T, \tau'')|.$$

   Besides, $|S_n(T, \tau)| = f_n$ for all $n \leq k-1$ (see Simion and Schmidt [6], Proposition 15), where f_n is the n-th Fibonacci number. Hence

   $$e_\tau(x) = \frac{x^2}{1-x}e_{q(\tau)}(x) + \frac{1}{1-x}.$$

Hence, by the definitions and Lemma 1, the theorem holds. ∎

Example 5. Let $T = \{123, 132, 213\}$. By Theorem 5,

1. $|S_n(T, 3412)| = n$.
2. $|S_n(T, 4231)| = |S_n(T, 3421)| = 4$.

3.2 T={123, 132, 231}.

Theorem 6. *Let $T = \{123, 132, 231\}$ and $\tau \in S_k(T)$. Then:*

(i) there exists r, $1 \leq r \leq k$, such that $\tau = (k, \dots, r+1, r-1, \dots, 1, r)$;
(ii) for all $n \geq k$

$$|S_n(T, (k, \dots, r+1, r-1, \dots, 1, r)| = k - 1,$$

where $2 \leq r \leq k$.

Proof. (i) Let $\tau \in S_k(T)$; put $r = \tau_n$. Since τ avoids 123, we see that τ contains $(r-1, \dots, 1, r)$, since τ avoids 132, we see that $\tau = (\tau_1, \dots, \tau_{k-r}, r-1, \dots, 1, r)$, and since τ avoids 231, we get that $\tau = (k, \dots, r+1, r-1, \dots, 1, r)$.

(ii) Let $\alpha \in S_n(T, \tau)$; similarly to (i), $\alpha = (n, \dots, t+1, t-1, \dots, 1, t))$ for $1 \leq t \leq n$, hence $|S_n(T, \tau)| = k - 1$. ■

Example 6. Let $T = \{123, 132, 231\}$; by Theorem 6,

$$|S_n(T, 4312)| = |S_n(T, 4213)| = |S_n(T, 3214)| = 3.$$

3.3 T={123, 213, 231}.

Theorem 7. *Let $T = \{123, 213, 231\}$ and $\tau \in S_k(T)$. Then:*

(i) there exists r, $1 \leq r \leq k$, such that $\tau = (k, \dots, r+1, 1, r, \dots, 2)$;
(ii) for all $n \geq k$

$$|S_n(T, (k, \dots, r+1, 1, r, \dots, 2))| = k - 1,$$

where $2 \leq r \leq k$.

Proof. (i) Let $\tau \in S_k(T)$ and choose r such that $\alpha_{k-r+1} = 1$. Since τ avoids 213, we get that $\tau_i > \tau_j$ for all $i < k-r+1 < j$, since τ avoids, 123 we see that τ contains $(1, r, \dots, 2)$, and since α avoids 231, we get that $\tau = (k, \dots, r+1, 1, r, \dots, 2)$.

(ii) Let $\alpha \in S_n(T, \tau)$; similarly to (i), $\alpha = (n, \dots, t+1, 1, t, \dots, 2)$ for $1 \leq t \leq n$, hence $|S_n(T, \tau)| = k - 1$. ■

Example 7. Let $T = \{123, 213, 231\}$; by Theorem 7,

$$|S_n(T, 4312)| = |S_n(T, 4132)| = |S_n(T, 1432)| = 3.$$

3.4 T={123, 231, 312}.

Theorem 8. *Let* $T = \{123, 231, 312\}$ *and* $\tau \in S_k(T)$. *Then :*

(i) there exists r, $1 \le r \le k$, *such that* $\tau = (r, \ldots, 2, 1, k, \ldots, r+1)$;
(ii) for all $n \ge k$
$$|S_n(T, (r, \ldots, 2, 1, k, \ldots, r+1))| = k - 1,$$
where $1 \le r \le k-1$.

Proof. (i) Let $\tau \in S_k(T)$; put $r = \tau_1$. Since τ avoids 123, we get that τ contains $(r, k, \ldots, r+1)$, and since τ avoids 231, we see that $\tau = (r, \ldots, k, \ldots, r+2, r+1)$, and since τ avoids 312, we get that $\tau = (r, \ldots, 2, 1, k, \ldots, r+2, r+1)$ for $r = 1, \ldots, k$.

(ii) Let $r \le k-1$ and $\alpha \in S_n(T, \tau)$; similarly to (i), $\alpha = (t, \ldots, 2, 1, n, \ldots, t+2, t+1)$ for $1 \le t \le n$. Hence $|S_n(T, \tau)| = k - 1$. ■

Example 8. Let $T = \{123, 231, 312\}$; by Theorem 8,
$$|S_n(T, 1432)| = |S_n(T, 2143)| = |S_n(T, 3214)| = 3.$$

3.5 T={132, 213, 231}.

Theorem 9. *Let* $T = \{132, 213, 231\}$ *and* $\tau \in S_k(T)$. *Then :*

(i) there exists r, $1 \le r \le k$, *such that* $\tau = (k, \ldots, r+1, 1, 2, \ldots, r)$;
(ii) for all $n \ge k$
$$|S_n(T, (k, \ldots, r+1, 1, 2, \ldots, r))| = k - 1.$$

Proof. (i) Let $\tau \in S_k(T)$; put $r = \tau_n$. Since τ avoids 231, we get that τ contains $(k, \ldots, r+1, r)$, since τ avoids 132, we see that $\tau = (k, \ldots, r+1, \tau_{k-r+1}, \ldots, \tau_{k-1}, r)$, and since τ avoids 213, we get that $\tau = (k, k-1, \ldots, r+1, 1, 2, \ldots, r)$ for $r = 1, \ldots, k$.

(ii) Let $\alpha \in S_n(T, \tau)$; similarly to (i), $\alpha = (n, \ldots, t+1, 1, 2, \ldots, t)$ for $1 \le t \le n$. Hence $|S_n(T, \tau)| = k - 1$. ■

Example 9. Let $T = \{132, 213, 231\}$; by Theorem 9,
$$|S_n(T, 4321)| = |S_n(T, 4312)| = |S_n(T, 4123)| = |S_n(T, 1234)| = 3.$$

Wilf class C	$\lvert C\rvert$	*Cardinality of $S_n(T)$, $T \in C$*	*Reference*
$\overline{\{123, 4321\}}$	490	0	Erdös and Szekeres [2]
$\overline{\{123, 1234\}}$	60	c_n	West [8], Knuth [5]
$\overline{\{123, 1432\}}$	46	f_{2n-2}	West [8]
$\overline{\{132, 3421\}}$	12	$1 + (n-1)2^{n-2}$	West [8] , Guibert [4]
$\overline{\{123, 2431\}}$	8	$3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$	West [8]
$\overline{\{123, 3421\}}$	4	$\binom{n}{4} + 2\binom{n}{3} + n$	West [8]
$\overline{\{132, 3214\}}$	4	Generating function $\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West [8]
$\overline{\{132, 4321\}}$	4	$\binom{n}{4} + \binom{n+1}{4} + \binom{n}{2} + 1$	West [8]
$\overline{\{123, 3412\}}$	2	$2^{n+1} - \binom{n+1}{3} - 2n - 1$	Billey, Jockusch and Stanley [1]
$\overline{\{123, 4231\}}$	2	$\binom{n}{5} + 2\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + 1$	West [8]
$\overline{\{123, 132, 1234\}}$	160	2^{n-1}	West [8], Simion and Schmidt [6]
$\overline{\{123, 132, 3412\}}$	118	$\binom{n}{2} + 1$	Section 2, Examples 1, 3, 4
$\overline{\{123, 312, 1432\}}$	24	$2n - 2$	Section 2, Example 2
$\overline{\{123, 132, 3241\}}$	12	$f_{n+2} - 1$	Section 2, Examples 1, 3
$\overline{\{123, 132, 3421\}}$	8	$3n - 5$	Section 2, Example 1
$\overline{\{123, 132, 3214\}}$	6	t_n	Section 2, Examples 1, 3
$\overline{\{123, 132, 231, 1234\}}$	282	n	West [8], Simion and Schmidt [6]
$\overline{\{123, 132, 231, 3214\}}$	46	3	Section 3, Examples 6, 7, 8, 9
$\overline{\{123, 132, 213, 1234\}}$	38	f_{n+1}	West [8], Simion and Schmidt [6]
$\overline{\{123, 132, 213, 3421\}}$	6	4	Section 3, Example 5
$\overline{\{123, 132, 213, 231, 1234\}}$	100	2	Section 4, Theorem 10
$\overline{\{123, 132, 213, 231, 4312\}}$	56	1	Section 4, Theorem 10

Table 1. Wilf classes of $\{T, \tau\}$, where $T \subseteq S_3$, $\tau \in S_4$

4 At least four patterns from S_3 and a pattern from S_k

By Simion and Schmidt [6], Proposition 17,

$$|S_n(T)| = 2,$$

where $\{123, 321\} \not\subset T \subset S_3$, $|T| = 4, 5$, and

$$|S_n(T)| = 0,$$

where $\{123, 321\} \subset T$. Hence, we obtain the following theorem.

Theorem 10. *Let $T \subset S_3$, $|T| \geq 4$ and $\tau \in S_k(T)$. For all $n \geq k$*

$$|S_n(T,\tau)| = \begin{cases} 2 - \delta_{\tau,(1,2,\dots,k)} - \delta_{\tau,(k,\dots,2,1)} & 123, 321 \notin T \\ 2 - \delta_{\tau,(k,\dots,2,1)} & 123 \in T,\ 321 \notin T \\ 2 - \delta_{\tau,(1,2,\dots,k)} & 123 \notin T,\ 321 \in T \\ 0 & 123, 321 \in T. \end{cases},$$

where $\delta_{x,y}$ the Kronecker symbol.

5 Wilf classes of $\{T, \tau\}$, where $T \subseteq S_3$, $\tau \in S_4$

By all the examples in all the sections we obtain Table 1. This table describes all the Wilf classes of sets of permutations avoiding a pattern from S_4 and a set of patterns from S_3. It contains 22 Wilf classes for sets $\{T, \tau\}$ where $T \subseteq S_3$, $\tau \in S_4$.

References

1. S. Billey, W. Jockusch et R.P. Stanley, Some combinatorial properties of Schubert polynomials, *Journal of Algebraic Combinatorics* 2 (1993) 345–374.
2. P. Erdös and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* 2 (1935) 463–470.
3. M. Feinberg, Fibonacci-Tribonacci, *Fib. Quart.* 1(#3) (1963), 71–74.
4. O. Guibert, Permutations sans sous séquence interdite, *Mémoire de Diplôme d'Etudes Aprrofondies de L'Université Brodeaux I* (1992).
5. D.E. Knuth, The art of computer programming, volume 1, Fundamental algorithms, Addison-Wesley, Reading, Massachusetts (1973).
6. R. Simion and F.W. Schmidt, Restricted permutations, *European Journal of Combinatorics* 6 (1985) 383–406.
7. Z.E. Stankova, Forbidden subsequences, *Discrete Mathematics* 132 (1994) 291–316.
8. J. West, Generating trees and forbidden subsequences, *Discrete Mathematics* 157 (1996) 363–372.